Horndeski Equations Summary

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What's in this document:

- The Lagrangian of Horndeski gravity and some notational preliminaries;
- The homogeneous background field equations;
- The full set of linearised field equations with $\alpha_H \neq 0$ (α_H is the 'Beyond Horndeski' parameter);
- The growth equation for Φ (+ associated constraint equation) for $\alpha_H = 0$, for both general and pressureless matter sectors;
- The quasistatic limit in terms of a modified Poisson and slip relation, a.k.a. $G_{\rm eff}$ and γ, for $\alpha_H = 0$.

1 Action and Notation

Two key resources for linear perturbation theory in Horndeski are:

- *Maximal freedom at minimum cost: linear large-scale structure in general modifications of gravity*, Emilio Bellini & Iggy Sawicki, https://arxiv.org/abs/1404.3713.
- *A unifying description of dark energy*, Jerome Gleyzes, David Langlois & Filippo Vernizzi, https://arxiv.org/abs/1411.3712v2.

The reason both are needed: the Bellini paper contains the first appearance of the Horndeski alpha parameters (see below), and uses the notation implemented in hiClass. However, at the time of its publication, the fifth 'Beyond Horndeski' parameter was not known. The Gleyzes paper contains the full set of equations including the fifth parameter. However, it has a few minor notational differences from Bellini, which (presumably) don't match hiClass.

Also, for the equations in §4, the Bellini paper assumes pressureless matter whilst the Gleyzes paper does not.

Specifically:

Bellini	**Gleyzes**
$X = -\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi$	$X = \nabla^\mu\phi\nabla_\mu\phi$
$-G_3$	G_3
G_{4X}	$-2G_{4X}$
$-\frac{1}{6}G_{5X}$	$\frac{1}{3}G_{5X}$
$-\alpha_B$	$2\alpha_B$
Φ	Ψ
Ψ	Φ
v_x	$-\pi$
$p_m\pi_m$	$-\sigma_m$

In the notation above, G_{4X} is shorthand for dG_4/dX. Hence most of the different factors of -2 above are the result of the different definitions of X in the first line. The penultimate entry is the variable that contains the scalar field perturbation, and the last entry is the anisotropic stress of matter.

In the Bellini notation, the Lagrangian is:

$$
\begin{aligned}
S &= \int \mathrm{d}^4x\,\sqrt{-g}\left[\sum_{i=2}^{5}\mathcal{L}_i + \mathcal{L}_{\rm m}[g_{\mu\nu}]\right],\\
\mathcal{L}_2 &= K(\phi,\,X)\,,\\
\mathcal{L}_3 &= -G_3(\phi,\,X)\Box\phi\,,\\
\mathcal{L}_4 &= G_4(\phi,\,X)R + G_{4X}(\phi,\,X)\left[(\Box\phi)^2 - \phi_{;\mu\nu}\phi^{;\mu\nu}\right],\\
\mathcal{L}_5 &= G_5(\phi,\,X)G_{\mu\nu}\phi^{;\mu\nu} - \frac{1}{6}G_{5X}(\phi,\,X)\left[(\Box\phi)^3 + 2{\phi_{;\mu}}^{\nu}{\phi_{;\nu}}^{\alpha}{\phi_{;\alpha}}^{\mu} - 3\phi_{;\mu\nu}\phi^{;\mu\nu}\Box\phi\right].
\end{aligned}
$$

In this note we will use the following form of the perturbed line element:

$$
ds^2 = -\left(1+2\Psi\right)dt^2 + a(t)^2\left(1-2\Phi\right)dx_i dx^i \tag{1}
$$

2 Background Equations

The effective Planck mass is given by (note, generally time-evolving):

$$
M_*^2 \equiv 2\left(G_4 - 2XG_{4X} + XG_{5\phi} - \dot{\phi}HXG_{5X}\right), \tag{2}
$$

where ϕ here is the homogeneous value of the scalar field, i.e. $\phi = \bar{\phi}$. Note that the above expression defines the parameter α_M through:

$$
\alpha_{\rm M} \equiv H^{-1}\frac{\mathrm{d}\ln M_*^2}{\mathrm{d}t}\,. \tag{3}
$$

The Friedmann equations are (dots denoting derivatives w.r.t. physical time):

$$
\begin{aligned}
3H^2 &= \frac{1}{M_*^2}\left[\rho_{\rm m} + \mathcal{E}\right] \qquad (4)\\
2\dot{H} + 3H^2 &= -\frac{1}{M_*^2}\left[p_{\rm m} + \mathcal{P}\right]
\end{aligned}
$$

where the effective energy density and pressure of the Horndeski sector are:

$$
\begin{aligned}
\mathcal{E} &\equiv -K + 2X\left(K_X - G_{3\phi}\right) + 6\dot{\phi}H\left(XG_{3X} - G_{4\phi} - 2XG_{4\phi X}\right) \qquad (5)\\
&\quad +12H^2X\left(G_{4X} + 2XG_{4XX} - G_{5\phi} - XG_{5\phi X}\right) + 4\dot{\phi}H^3X\left(G_{5X} + XG_{5XX}\right),\\
\mathcal{P} &= K - 2X\left(G_{3\phi} - 2G_{4\phi\phi}\right) + 4\dot{\phi}H\left(G_{4\phi} - 2XG_{4\phi X} + XG_{5\phi\phi}\right) \qquad (6)\\
&\quad -M_*^2\alpha_{\rm B}H\frac{\ddot{\phi}}{\dot{\phi}} - 4H^2X^2G_{5\phi X} + 2\dot{\phi}H^3XG_{5X}\,.
\end{aligned}
$$

But I suppose the above are not needed if one wishes to parameterise in the usual e.o.s. way, $w_X = \mathcal{P}/\mathcal{E}$.

Note that ρ_m obeys its usual conservation law, BUT the quantity (ρ_m/M_*^2) does not, due to the time-dependence of the denominator. So beware which is implied/implemented in a code. In other contexts, we sometimes work in Planck units and hence hide the M_*^2; that can't be done here.

3 Perturbation Equations (Full Set)

Here I have taken the linearised field equations from Gleyzes, but converted them to Bellini notation, as per §1. v_x is the (normalised) perturbation of the scalar field, $v_x = -\delta\phi/\dot{\phi}$. (Not to be confused/related in any way to v_m, the velocity perturbation of the matter sector.)

There are four principal 'alpha' parameters; α_M is given by eqs.(2) and (3) above, and the remaining three are related to the G_i functions appearing in the Horndeski Lagrangian by:

$$\begin{aligned} H^2 M_*^2 \alpha_\mathrm{K} = {} & 2X\left(K_X + 2XK_{XX} - 2G_{3\phi} - 2XG_{3\phi X}\right) + \\ & + 12\dot\phi XH\left(G_{3X} + XG_{3XX} - 3G_{4\phi X} - 2XG_{4\phi XX}\right) + \\ & + 12XH^2\left(G_{4X} + 8XG_{4XX} + 4X^2G_{4XXX}\right) - \\ & - 12XH^2\left(G_{5\phi} + 5XG_{5\phi X} + 2X^2G_{5\phi XX}\right) + \\ & + 4\dot\phi XH^3\left(3G_{5X} + 7XG_{5XX} + 2X^2G_{5XXX}\right) \end{aligned} \tag{7}$$

$$\begin{aligned} HM_*^2\alpha_\mathrm{B} = {} & 2\dot\phi\left(XG_{3X} - G_{4\phi} - 2XG_{4\phi X}\right) + \\ & + 8XH\left(G_{4X} + 2XG_{4XX} - G_{5\phi} - XG_{5\phi X}\right) + \\ & + 2\dot\phi XH^2\left(3G_{5X} + 2XG_{5XX}\right) \end{aligned} \tag{8}$$

$$M_*^2\alpha_\mathrm{T} = 2X\left(2G_{4X} - 2G_{5\phi} - \left(\ddot\phi - \dot\phi H\right)G_{5X}\right) \tag{9}$$

These alpha parameters are more closely linked to observations and physical effects than the G_i's in the Lagrangian. Hence they are the things to be focussed on. **EDIT: These expressions can be simplified in light of the results from GW170817. The simplest interpretation of these sets $\alpha_T = 0$, or equivalently, $G_{4X} = G_5 = 0$ (and by implication their derivatives, e.g. G_{4XX}, also equal to zero).**

There is also a fifth alpha, the Beyond Horndeski parameter α_H, which appears in the first set of equations below. This is present if additional disformal terms are added to the Lagrangian in eq.(1). I won't delve in to this complication here, though I can do the necessary legwork if we decide that this is something we want to pursue in future. Full details can be found in the Gleyzes paper referenced above.

The linearised field equations are:

$$\begin{aligned} 00: \quad & 3(2-\alpha_B)H\dot\Phi + (6-\alpha_K-6\alpha_B)H^2\Psi + 2(1+\alpha_H)\frac{k^2}{a^2}\Phi \\ & - (\alpha_K + 3\alpha_B)H^2\dot v_x - 6\left[(1-\frac{\alpha_B}{2})\dot H + \frac{\rho_\mathrm{m}+p_\mathrm{m}}{2{M_*}^2} + \frac{1}{3}\frac{k^2}{a^2}\left(\alpha_H + \frac{\alpha_B}{2}\right)\right]Hv_x = -\frac{\delta\rho_\mathrm{m}}{{M_*}^2}\,, \end{aligned} \tag{10}$$

$$0i: \quad 2\dot\Phi + (2-\alpha_B)H\Psi - H\alpha_B\dot v_x - \left(2\dot H + \frac{\rho_\mathrm{m}+p_\mathrm{m}}{{M_*}^2}\right)v_x = -\frac{(\rho_\mathrm{m}+p_\mathrm{m})v_\mathrm{m}}{{M_*}^2}\,. \tag{11}$$

$$ij \text{ traceless}: \quad (1+\alpha_H)\Psi - (1+\alpha_T)\Phi - (\alpha_M - \alpha_T)Hv_x + \alpha_H\dot v_x = \frac{p_\mathrm{m}\pi_\mathrm{m}}{{M_*}^2}\,, \tag{12}$$

$$\begin{aligned} \text{trace}: \quad & 2\ddot\Phi + 2(3+\alpha_M)H\dot\Phi + (2-\alpha_B)H\dot\Psi \\ & + 2\left[\dot H - \frac{\rho_\mathrm{m}+p_\mathrm{m}}{2{M_*}^2} - \frac{1}{2}(\alpha_B H)^\cdot + (3+\alpha_M)\left(1\frac{1}{2}\alpha_B\right)H^2\right]\Psi \\ & - H\alpha_B\,\ddot v_x + 2\left[\dot H + \frac{\rho_\mathrm{m}+p_\mathrm{m}}{2{M_*}^2} + \frac{1}{2}(\alpha_B H)^\cdot + (3+\alpha_M)\frac{\alpha_B}{2}H^2\right]\dot v_x \\ & - 2\left[(3+\alpha_M)H\dot H + \frac{\dot p_\mathrm{m}}{2{M_*}^2} + \ddot H\right]v_x = \frac{1}{{M_*}^2}\left(\delta p_\mathrm{m} + \frac{2}{3}\frac{k^2}{a^2}(p_m\pi_\mathrm{m})\right)\,. \end{aligned} \tag{13}$$

The evolution equation for π reads

$$H^2\alpha_K\ddot v_x + \left\{\left[H^2(3+\alpha_M) + \dot H\right]\alpha_K + (H\alpha_K)^\cdot\right\}H\dot v_x \tag{14}$$

$$+ 3\left\{\left(\dot H + \frac{\rho_\mathrm{m}+p_\mathrm{m}}{2{M_*}^2}\right)2\dot H - \dot H\alpha_B\left[H^2(3+\alpha_M) + \dot H\right] - H(\dot H\alpha_B)^\cdot\right\}v_x \tag{15}$$

$$- 2\frac{k^2}{a^2}\left\{\dot H + \frac{\rho_\mathrm{m}+p_\mathrm{m}}{2{M_*}^2} + H^2\left[1 - \frac{1}{2}\alpha_B(1+\alpha_M) + \alpha_T - (1+\alpha_H)(1+\alpha_M)\right] - \left(H\left(\frac{\alpha_B}{2} + \alpha_H\right)\right)^\cdot\right\}v_x \tag{16}$$

$$+ 3H\alpha_B\ddot\Phi + H^2(3\alpha_B + \alpha_K)\dot\Psi - 3\left[2\dot H + \frac{\rho_\mathrm{m}+p_\mathrm{m}}{{M_*}^2} - H^2\alpha_B(3+\alpha_M) - (\alpha_B H)^\cdot\right]\dot\Phi \tag{17}$$

$$- \left[6\left(\dot H + \frac{\rho_\mathrm{m}+p_\mathrm{m}}{2{M_*}^2}\right) - H^2(3\alpha_B + \alpha_K)(3+\alpha_M) - (9\alpha_B + 2\alpha_K)\dot H - H(3\dot\alpha_B + \dot\alpha_K)\right]H\Psi \tag{18}$$

$$- 2\frac{k^2}{a^2}\left\{\alpha_H\dot\Phi + \left[H(\alpha_M + \alpha_H(1+\alpha_M) - \alpha_T) - \dot\alpha_H\right]\Phi - \left(\alpha_H + \frac{1}{2}\alpha_B\right)H\Psi\right\} = 0\,. \tag{19}$$

4 Growth System

4.1 General Matter Sector

The full set of field equations above is somewhat messy. A somewhat more convenient system to solve can be formed by eliminating the scalar field perturbation from eqs.(10), (11) and (13) (involves taking a time derivative of eq.13). The result is a second order DE for the potential Φ, sourced by δ_m, see below Note that **both Bellini & Gleyzes papers set $\alpha_H = 0$ at this point**.

$$\ddot{\Phi} + \frac{\beta_1\beta_2 + \beta_3\alpha_B^2\,\tilde{k}^2}{\beta_1 + \alpha_B^2\tilde{k}^2} H\dot{\Phi} + \frac{\beta_1\beta_4 + \beta_1\beta_5\,\tilde{k}^2 + c_s^2\alpha_B^2\tilde{k}^4}{\beta_1 + \alpha_B^2\,\tilde{k}^2} H^2\Phi = -\frac{1}{2M_*^2}\Bigg[\frac{\beta_1\beta_6 + \beta_7\alpha_B^2\,\tilde{k}^2}{\beta_1 + \alpha_B^2\,\tilde{k}^2}\delta\rho_{\mathrm{m}}$$
$$+ \frac{\beta_1\beta_8 + \beta_9\alpha_B^2\,\tilde{k}^2}{\beta_1 + \alpha_B^2\,\tilde{k}^2} H(\rho_m + p_m)v_m - \frac{\beta_1\beta_{10} + \beta_1\beta_{11}\,\tilde{k}^2 + \frac{2}{3}\alpha_B^2\tilde{k}^4}{\beta_1 + \alpha_B^2\,\tilde{k}^2} H^2 p_m\pi_m - \frac{\alpha_K}{\alpha}\delta p_{\mathrm{m}} + 2H\frac{d}{dt}(p_m\pi_m)\Bigg] , \tag{20}$$

Here the functions $\beta_i \equiv \beta_i\,(t)$ are combinations of the α_i functions, given below, and $\tilde{\mathbf{k}} = \mathbf{k}/\mathbf{aH}$. By eliminating variables again, one can rewrite the slip relation as:

$$\alpha_B^2\tilde{k}^2\left[\Psi - \Phi\left(1 + \alpha_T - \frac{4\gamma_9}{\alpha\alpha_B}\right) - \frac{p_m\pi_m}{M_*^2}\right] + \beta_1\left[\Psi - \Phi(1+\alpha_T)\frac{\gamma_1}{\beta_1} - \frac{p_m\pi_m}{M_*^2}\right] =$$
$$\frac{4\gamma_9}{H^2M_*^2}\left[-\frac{\alpha_B}{2\alpha}\left(\delta\rho_{\mathrm{m}} - 3H(\rho_m + p_m)v_m\right) + HM_*^2\,\dot{\Phi} + H\frac{\alpha_K}{2\alpha}\,q_{\mathrm{m}} + H^2\left(p_m\pi_m\right)\right] . \tag{21}$$

I am confused about the appearance of γ_9 above – potentially it could be a typo in the Gleyzes paper. If this turns out to be really crucial, I can investigate further. However, I suspect the simplified version of these equations (see below) are the ones we might actually want to use.

$$\beta_1 = -\alpha_K\frac{\rho_{\mathrm{m}} + p_{\mathrm{m}}}{H^2M_*^2} - 2\alpha\left(\frac{\dot{H}}{H^2} + \alpha_T - \alpha_M\right) , \tag{22}$$

$$\beta_2 \equiv 2(2 + \alpha_M) + 3\Upsilon , \tag{23}$$

$$\beta_3 \equiv 3 + \alpha_M + \frac{\alpha_B^2}{H\alpha}\left(\frac{\alpha_K}{\alpha_B^2}\right)^{\cdot} , \tag{24}$$

$$\beta_4 \equiv (1+\alpha_T)\left[2\dot{H}/H^2 + 3(1+\Upsilon) + \alpha_M\right] + \dot{\alpha}_T/H , \tag{25}$$

$$\beta_5 \equiv c_s^2 - \frac{2\alpha_B(\beta_3 - \beta_2)}{\alpha} + \frac{\alpha_B^2}{4\beta_1}(1+\alpha_T)(\beta_3 - \beta_2) + \frac{\alpha_B^2\beta_4}{4\beta_1} , \tag{26}$$

$$\beta_6 \equiv \beta_7 + \frac{\alpha_B(\beta_3 - \beta_2)}{\alpha} , \tag{27}$$

$$\beta_7 \equiv c_s^2 + \frac{\alpha_B^2/2(1+\alpha_T) - \alpha_B(\alpha_T - \alpha_M)}{\alpha} , \tag{28}$$

$$\beta_8 \equiv \beta_9 - \frac{(\alpha_K + 3\alpha_B)(\beta_3 - \beta_2)}{\alpha} , \tag{29}$$

$$\beta_9 \equiv -(1 + 3c_s^2 + \alpha_T) + \frac{\alpha_B^2}{H\alpha}\left(\frac{\alpha_K}{\alpha_B^2}\right)^{\cdot} , \tag{30}$$

$$\beta_{10} \equiv -6(1+\Upsilon) - 4\dot{H}/H^2 , \tag{31}$$

$$\beta_{11} \equiv \frac{2}{3} - \frac{\alpha_B^2}{2\beta_1}\left[(2 - \alpha_M) + 2\dot{H}/H^2\right] - \frac{\alpha_B^4}{2\beta_1H\alpha}\left(\frac{\alpha_K}{\alpha_B^2}\right)^{\cdot} \tag{32}$$

$$\gamma_1 \equiv \alpha_K\frac{\rho_D + p_D}{4H^2M_*^2} - 3\alpha_B^2\frac{\dot{H}}{H^2} , \tag{33}$$

$$\gamma_9 \equiv \alpha\frac{\alpha_T - \alpha_M}{2} . \tag{34}$$

with

$$\alpha = \alpha_K + \frac{3}{2}\alpha_B^2 \tag{35}$$

$$12\beta_1 H^3 M_*^2 \Upsilon \equiv \; 2\alpha M_*^2 \Big\{ \big[\dot{H} + (\alpha_T - \alpha_M)H^2\big]^{\cdot} + (3+\alpha_M)H\big[\dot{H} + (\alpha_T - \alpha_M)H^2\big] \Big\} \tag{36}$$

$$+ \alpha_K \dot{p}_{\rm m} - (\rho_{\rm m} + p_{\rm m})H(\alpha_K + 3\alpha_B)(\alpha_T - \alpha_M) + \frac{3}{2}(\rho_{\rm m} + p_{\rm m})\frac{\alpha_B^4}{\alpha}\left(\frac{\alpha_K}{\alpha_B^2}\right)^{\cdot} \tag{37}$$

$$c_s^2 = -\frac{(2-\alpha_B)\Big[\dot{H} - (\alpha_M - \alpha_T)H^2 - H^2\alpha_B/2(1+\alpha_T)\Big] - H\dot{\alpha}_B + (\rho_{\rm m} + p_{\rm m})/M_*^2}{H^2\alpha} \tag{38}$$

4.2 Pressureless dust matter sector

Eqs.(20) and (21) are still pretty ugly. The matter source terms, at least, can be somewhat simplified if we restrict to the case of pressureless matter, setting $p_m = \pi_m = 0$. Furthermore, Bellini et al. argue that the velocity perturbation v_m can be neglected on subhorizon scales. Implementing these simplifications:

$$\ddot{\Phi} + \frac{\beta_1\beta_2 + \beta_3\alpha_B^2\,\tilde{k}^2}{\beta_1 + \alpha_B^2\tilde{k}^2}H\dot{\Phi} + \frac{\beta_1\beta_4 + \beta_1\beta_5\,\tilde{k}^2 + c_s^2\alpha_B^2\tilde{k}^4}{\beta_1 + \alpha_B^2\,\tilde{k}^2}H^2\Phi = -\frac{1}{2M_*^2}\left[\frac{\beta_1\beta_6 + \beta_7\alpha_B^2\,\tilde{k}^2}{\beta_1 + \alpha_B^2\,\tilde{k}^2}\delta\rho_{\rm m}\right], \tag{39}$$

$$\alpha_{\rm B}^2\frac{k^2}{a^2}\left[\Psi - \Phi\left(1 + \alpha_{\rm T} + \frac{2\,(\alpha_{\rm M} - \alpha_{\rm T})}{\alpha_{\rm B}}\right)\right] + \beta_1\left[\Psi - \Phi\,(1+\alpha_{\rm T})\left(1 - \frac{2\alpha H^2\,(\alpha_{\rm M} - \alpha_{\rm T})}{\beta_1}\right)\right]$$
$$= (\alpha_{\rm M} - \alpha_{\rm T})\left[\alpha_{\rm B}\frac{\rho_{\rm m}\delta_{\rm m}}{M_*^2} - 2H\alpha\,\dot{\Phi}\right]. \tag{40}$$

The system is closed by the standard evolution equations for CDM:

$$\dot{\delta}_{\rm m} - \frac{k^2}{a^2}v_{\rm m} = 3\dot{\Phi}\,, \qquad \dot{v}_{\rm m} = -\Psi\,, \tag{41}$$

Eqs.(39) - (41) are then what one might code up to solve for the evolution of δ_m, from which the growth rate could be computed.

Alternatively, I guess eq.(39) could be rewritten as a second order DE for δ_m instead of Φ, with a little work. Let me know if this is something that we would really like to see.

5 Quasistatic Limit

There are two methods of taking the quasistatic limit, which only seem to agree in the extreme $k \to \infty$, scale-independent limit. Note that less extreme limits retaining some k-dependence have also been studied *(TB: more work needed to fill in details here)*.

There is also a choice of whether to parameterise the Poisson equation for Φ (which is what drops out of the Einstein equations when the line element in eq.1 is used), or the Poisson equation for Ψ (since this is the potential that controls growth of dust perturbations in GR). I will follow Gleyzes route here, and parameterise the potential that controls growth (Ψ):

So, the quasistatic (QS) parameterisation functions we'll use are, in Fourier space:

$$-2\frac{k^2}{a^2}\Psi = 8\pi G_{\rm eff}(z)\rho_m\delta_m$$
$$= -2\frac{k^2}{a^2}\Psi_{GR} \times \frac{G_{\rm eff}(z)}{G_N} \tag{42}$$

for the growth potential; note that in the QS limit $\Delta_m \approxeq \delta_m$, and so this is what appears on the RHS.

We use the standard definition of slip:

$$\gamma(z) = \frac{\Phi}{\Psi} \tag{43}$$

and the lensing parameter:

$$-\frac{k^2}{a^2}(\Phi+\Psi) = -\frac{k^2}{a^2}\Psi\left[1+\gamma(z)\right]$$
$$= -\frac{k^2}{a^2}\Psi_{GR}\,\frac{G_{\rm eff}(z)}{G_N}\left[1+\gamma(z)\right]$$
$$= -\frac{k^2}{a^2}\Psi_{GR}\times 2\Sigma(z) \tag{44}$$
$$\Rightarrow \Sigma(z) = \frac{1}{2}\frac{G_{\rm eff}(z)}{G_N}\left[1+\gamma(z)\right] \tag{45}$$

Note that Σ includes a factor of $1/2$ in its definition, such that the GR limit is $\Sigma = 1$. Then $G_{\rm eff}$ is given by:

$$\frac{G_{\rm eff}}{G_N} = \left(\frac{M_P^2}{M_*^2}\right)\frac{\alpha\, c_s^2(1+\alpha_T) + 2\left[-\alpha_B/2(1+\alpha_T)+\alpha_T-\alpha_M\right]^2}{\alpha\, c_s^2}\ , \tag{46}$$

where M_P is the usual Planck mass, and α and c_s^2 are defined in eqs.(35) and (38) above. The slip parameter is:

$$\gamma = \frac{\alpha\, c_s^2 - \alpha_B\left[-\alpha_B/2(1+\alpha_T)+\alpha_T-\alpha_M\right]}{\alpha\, c_s^2(1+\alpha_T) + 2\left[-\alpha_B/2(1+\alpha_T)+\alpha_T-\alpha_M\right]^2}\ , \tag{47}$$

Note that α_K does not feature in either of these, and hence is (effectively) impossible to constrain with QS data. α_H does not appear because we are still working with it switched off, as per section 4.

Σ can be straightforwardly found by using the above expressions and eq.(45); the resulting expression is not particularly insightful.